METRIC SPACES: FINAL EXAM 2014

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Evaluation: $\min\Big(100\%, \max(5\,\text{prb} \times 20\% \cdot \left[\begin{smallmatrix}1.00\\1.15^{\text{top}}\end{smallmatrix}\right],\ \sum_{i=1}^{6} \text{h/w} \times 5\% + 5\,\text{prb} \times 14\% \cdot \left[\begin{smallmatrix}1.00\\1.15^{\text{top}}\end{smallmatrix}\right])\Big)$.

Problem 1. For all $x, y \in \mathbb{R}$ put $\mathrm{d}(x,y) = \dfrac{|x-y|}{1+|x-y|}$ by definition. Prove that the function $\mathrm{d}\colon \mathbb{R} \times \mathbb{R} \to [0,1)$ is a metric on $\mathbb{R}$.

Problem 2. Let $(\mathcal{X}, \mathrm{d}_{\mathcal{X}})$ be a metric space and $S \subseteq \mathcal{X}$ be a bounded subset in it. Prove that the closure $\overline{S}$ is bounded and $\operatorname{diam} \overline{S} = \operatorname{diam} S$.

(By definition, $\operatorname{diam}(\varnothing) = 0$ and $\operatorname{diam}(S) = \sup_{x,y \in S} \mathrm{d}_{\mathcal{X}}(x,y)$ for a non-empty bounded set $S \subseteq \mathcal{X}$.)

Problem 3. Let $(\mathcal{X}, \mathrm{d}_{\mathcal{X}})$ be a metric space and $\{A_i \mid i \in \mathcal{I}\}$ be a family of connected subsets $A_i \subseteq \mathcal{X}$ such that $A_i \cap A_j \neq \varnothing$ for all indexes $i, j \in \mathcal{I}$. Prove that the union $A = \bigcup_{i \in \mathcal{I}} A_i$ is connected.

Problem 4. Suppose for every $n \in \mathbb{N}$ that V_n is a non-empty closed subset of a sequentially compact space $\mathcal{X}$ and $V_n \supseteq V_{n+1}$. Prove that

$$\bigcap_{n=1}^{+\infty} V_n \neq \varnothing.$$

- Is this intersection always non-empty if the hypothesis of sequential compactness is discarded? (state and prove, e.g., by counterexample)

Problem 5. Let $(\mathcal{X}, \mathrm{d}_{\mathcal{X}})$ be a non-empty complete metric space. Suppose that $f, g\colon \mathcal{X} \to \mathcal{X}$ are two Banach's contractions of $\mathcal{X}$. Prove that there always exists a unique point $x_0 \in \mathcal{X}$ such that $f\big(g(x_0)\big) = x_0$.

Date: April 3, 2014.

Do not postpone your success until May Day. GOOD LUCK!